Immersed boundary methods for numerical simulation of confined fluid and plasma turbulence in complex geometries

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Acknowledgements

Outline

Motivation

The volume penalization method for fixed (~~and moving~~) obstacles

Analysis of the penalized Laplace operator

1d example for Dirichlet and Neumann boundary conditions

Applications to fluid turbulence (Navier-Stokes eq.) :

2d confined turbulence, flapping wings in 3d

Application to passive scalars (turbulent mixing(ng)]TJ(b9aJET EMC

Context: Immersed boundary methods

spectral/wavelet solvers

ality term

+ pena

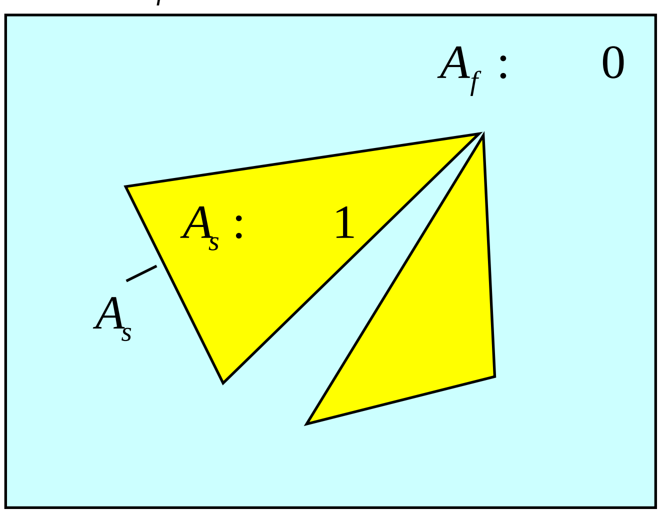
Physically motivated mathematical model

- Solid moving wings
- Viscous incompressible fluid

$$\nabla \cdot \mathbf{u} \quad 0, \mathbf{u} \qquad \overline{\quad} \qquad {}^{2}\mathbf{u}$$

$$\mathbf{u}| \qquad \mathbf{u}_s|$$

$A \quad A_f \cup A_s$



Penalized equation:

$$_t\mathbf{u} \quad \mathbf{u} \quad \mathbf{u} \quad \frac{1}{\quad} p \qquad {}^{2}\mathbf{u} \quad \frac{}{}(\mathbf{u} \quad \mathbf{u}_s) \quad 0$$

Volume penalization method is physically motivated

E. Arquis and J.P. Caltagirone, 1984.
C. R. Acad. Sci. Paris, **II**

Mathematically justified

P. Angot, C.H. Bruneau and P. Fabrie, 1999.
Numer. Math. **81**

G. Carbou and P. Fabrie, 2003.
Adv. Diff. Equations **8**

- Easy-to-implement

K. Schneider, 2005. *Comput. Fluids* **34**

Hydrodynamic force

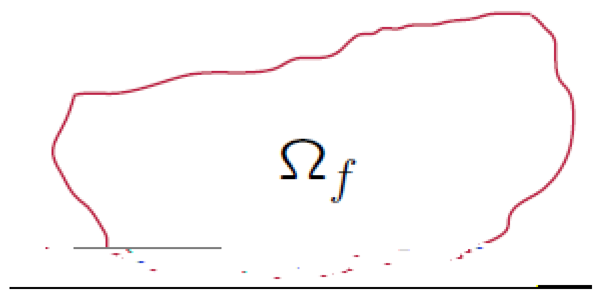
$$\mathbf{F} / \qquad \int_A \frac{}{}(\mathbf{u} \quad \mathbf{u}_s) dA \quad V_c \frac{d\mathbf{u}_c}{dt}$$

$\mathbf{u}_s$ pointwise velocity of the solid
$\mathbf{u}_c$ velocity of the center of the solid
V_c volume of the solid

Initial Boundary Value Problem
In complex geometry

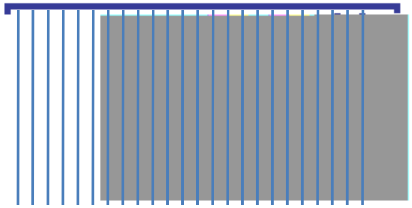
[redacted]

(Plus initial conditions)



with L being, e.g. the Laplace operator, or Navier-Stokes or Maxwell operator

Discretized penalized problem



with $\Delta x \propto 1/N$

Some analysis: a simple example

$$-\frac{d^2 w(x)}{dx^2} = f(x)$$

Dirichlet boundary conditions

the right hand side given by a sinusoidal function ... and

$f(x) = m^2 \sin$ ...

The exact solution to this problem is

$$w(x) = \sin mx$$

Exact solution of the penalized 1d Poisson equation

Penalization term

penalization term

$$-u'' + \frac{1}{\eta}\chi u = f$$

$$\chi = \begin{cases} 0, & x \in]0,\pi[\\ 1/2, & x = 0,\ x = \pi \\ 1, & x \in]\pi, 2\pi[\end{cases} \qquad (6)$$

penalized problem is

Penalization error of the Dirichlet problem

Discretization error of the penalized equation

The penalized problem is discretized with a pseudospectral Fourier method using N grid points. For the L^2 error between the discrete solution and the exact solution of the penalized problem we get,

where K=2 for m even and K 3.84 for m odd.
The N^{-2} behavior is related to the regularity the exact

[redacted]

How to choose ?

Combining the two estimates we get a bound for the total error e between the discrete-penalized solution and the exact solution of the Dirichlet problem:

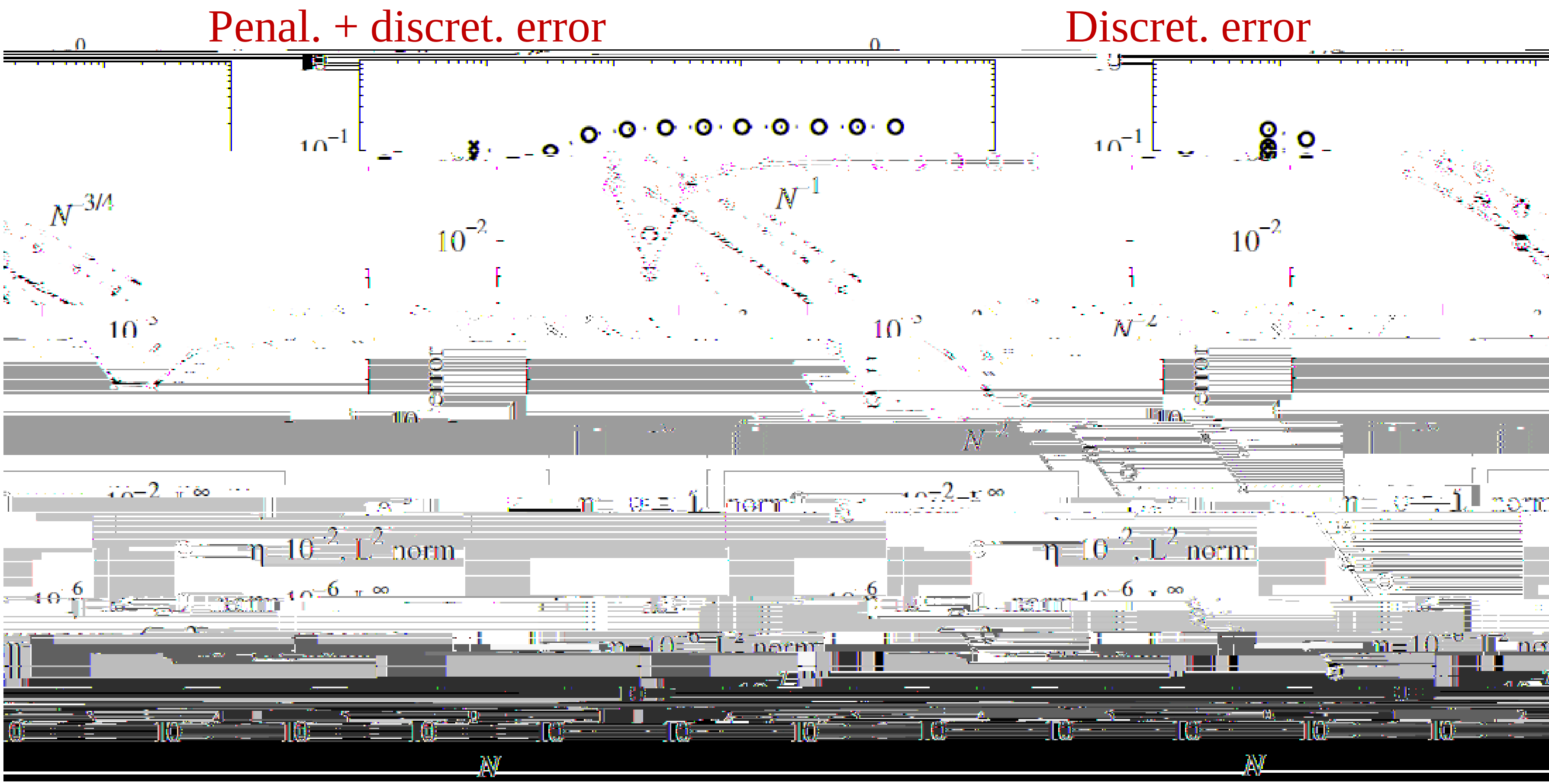
[illegible]

When is chosen with the right order of magnitude, i.e. $1/N$, in order to optimize the preceding estimate, then the resulting error is

$$\varepsilon \propto \frac{1}{N}$$

which suggests that the penalization method with Fourier discretization is a first order method.

Convergence of the Fourier collocation method



Error with respect to the exact Dirichlet solution in the interior of the fluid domain (left) and with respect to the penalized solution in the whole domain (right).

The volume penalization method for fixed (~~and moving~~) obstacles

Navier-Stokes equations

Two- and three-dimensional formulations

Two-dimensional model:

$$\quad {}^{2} \quad -(\quad {}_{s}) \quad 0$$

$$t$$

$$2$$

$$\mathbf{u} \qquad \mathbf{U}$$

Three-dimensional model:

$$\ _t\mathbf{u} \quad {}^{2} \quad -(\mathbf{u} \quad \mathbf{u}_s) \quad 0$$

$$^{2} \quad (\quad -(\mathbf{u} \quad \mathbf{u}_s)) \quad 0$$

$$p \quad u^2/2$$

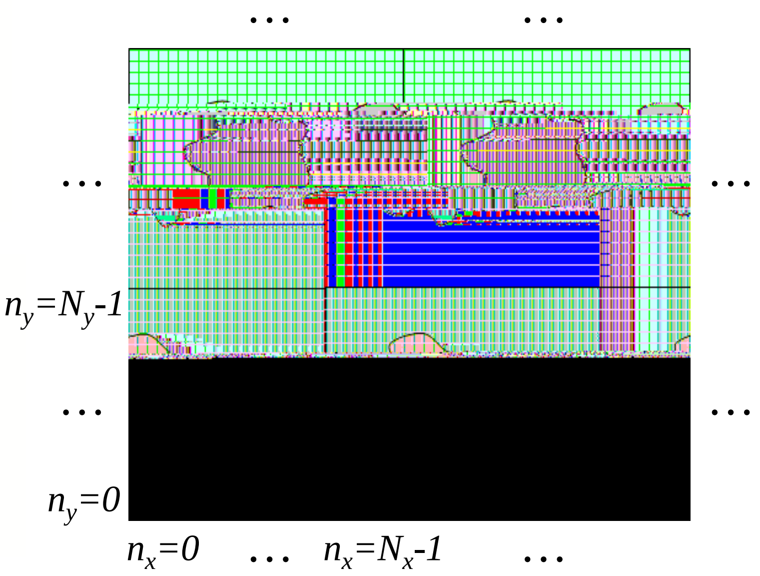
Numerical method

Pseudo-spectral Fourier discretization in space (periodic boundary conditions)
Fast Fourier Transform

Exact integration of the viscous term (method of integrating factors)

$_t \quad |\mathbf{k}|^2 \quad N(\quad)$

Adaptive 2nd order Adams-Bashforth time-stepping scheme

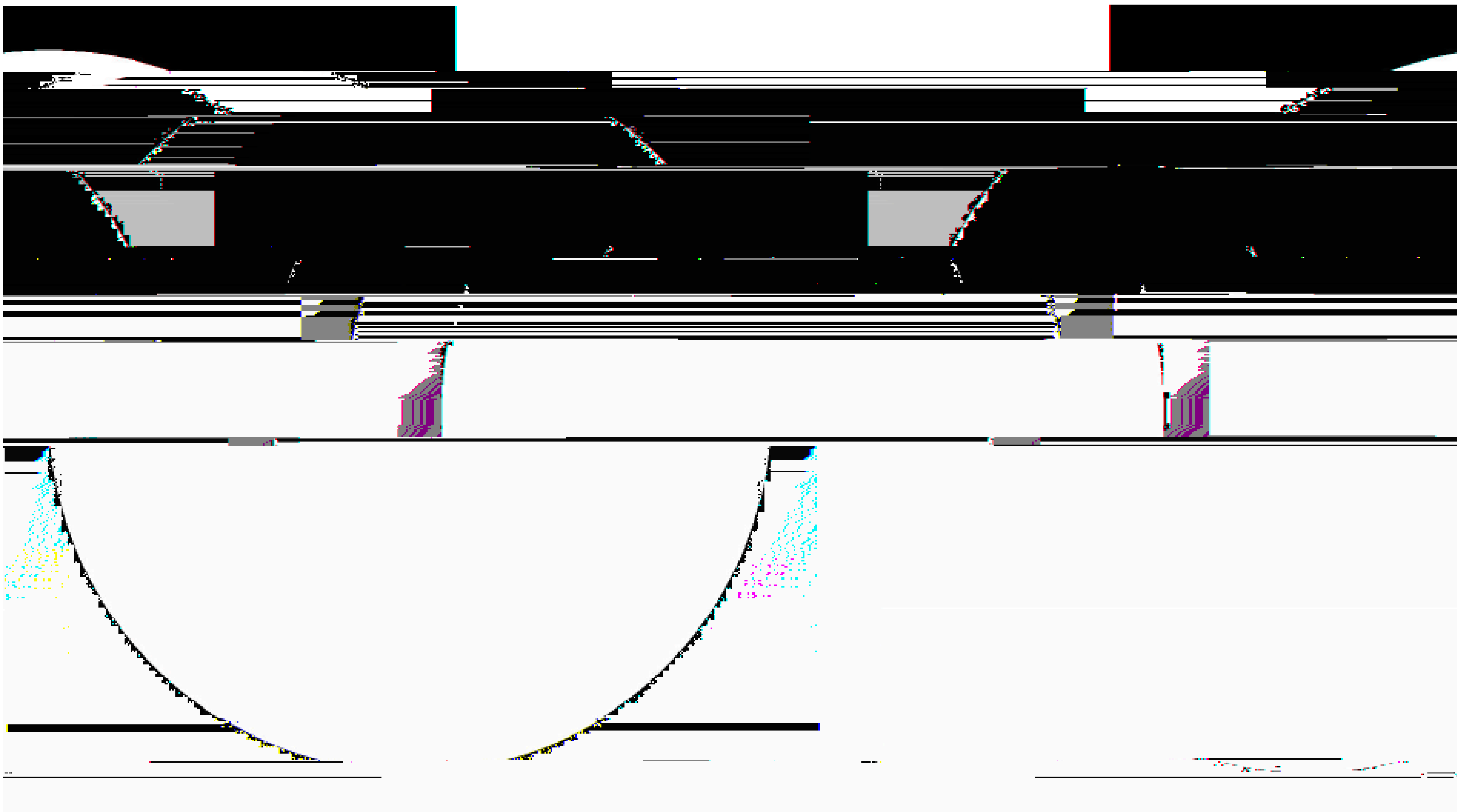


Ref.: K. Schneider, 2005. *Comput. Fluids* **34**
D. Kolomenskiy and K. Schneider, 2009. *J. Comput. Phys.* **228**

Application to

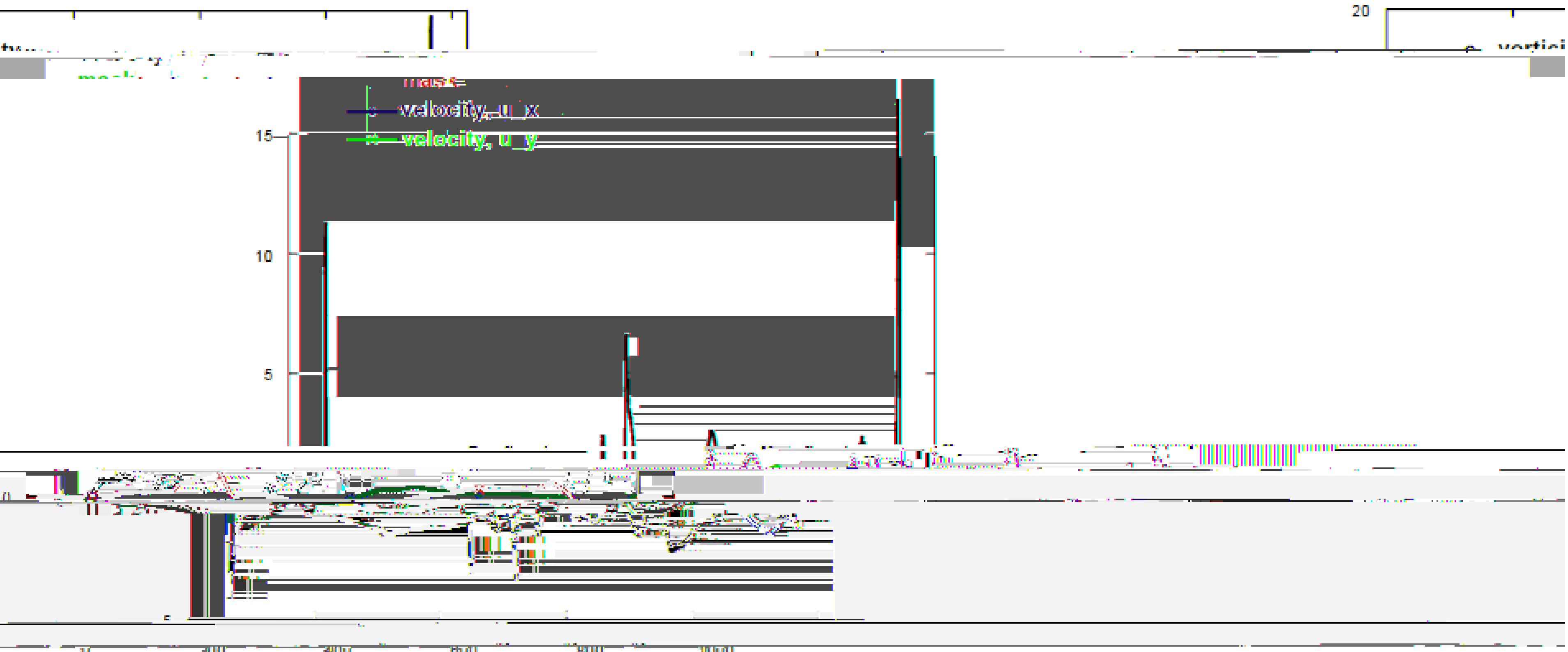
2d confined turbulence

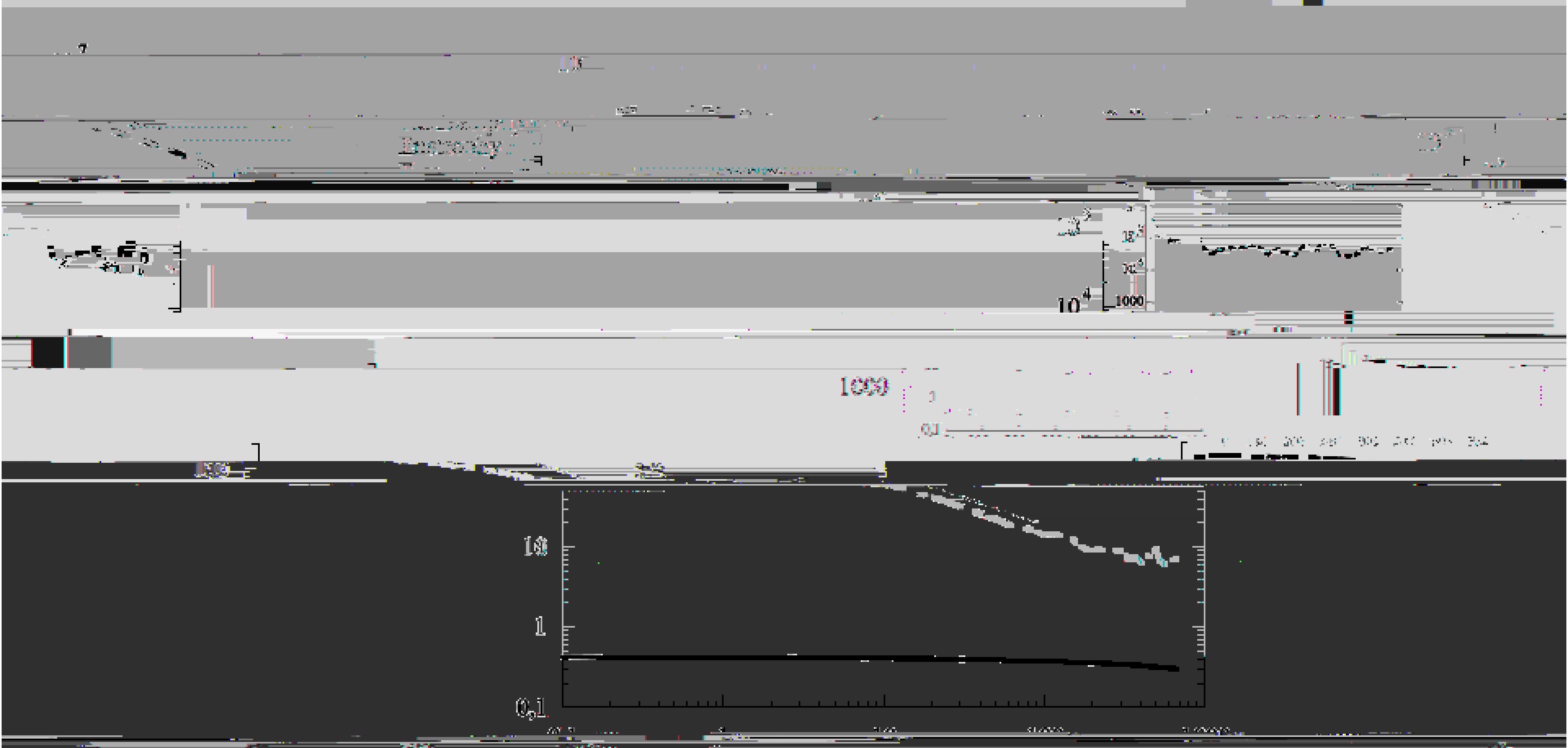
Mask:




0
Streamfunction
200
400

Vertical cuts at ix=511





Time evolution of energy E, enstrophy Z and palinstrophy P.

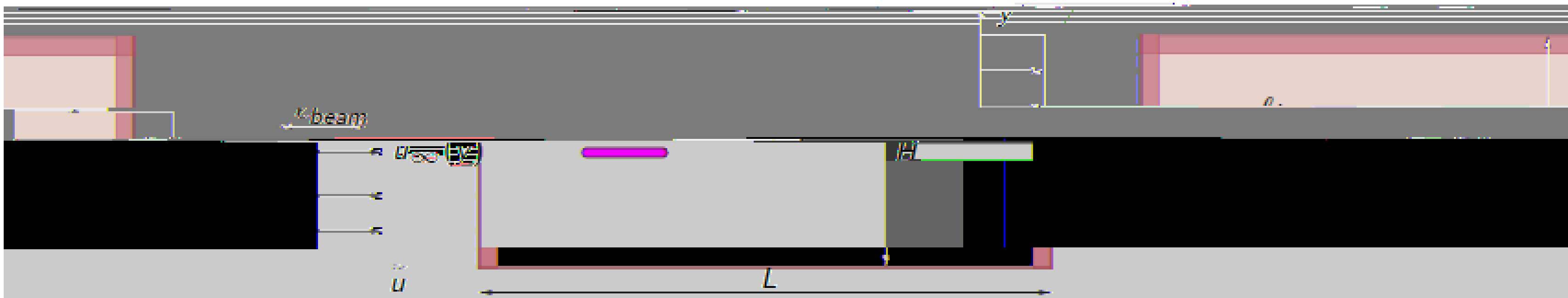
2D decaying turbulence in a circular domain

Flapping wings   Realistic planform and kinematics

www.vapor.ucar.edu

Ref.: D. Kolomenskiy, H.K. Moffatt, M. Farge, K. Schneider. *J. Fluids Struc.*, 2011

Penalization for Fluid structure interaction: flexible beam



Vorticity, Re=200, $=10^{-3}$ (chaotic state)

Ref.: Th. Engels, D. Kolomenskiy, K. Schneider and J. Sesterhenn, *Comput*. & *Struc*. 122, 2013.

Swimming

Neumann boundary conditions (I)

As simple example we consider the Poisson equation,

$$u''(x) = f(x), \quad x \in (0, \pi)$$

$$u'(x = 0) = u'(x = \pi) = 0$$

$$f(x) = m^2 \cos mx, \quad m \in \mathbb{Z}$$

[redacted]

[redacted]

[redacted]

Neumann boundary conditions (III)

Advection-diffusion equation of a passive scalar,

[illegible]

[illegible], [illegible]

and [redacted]

The penalization term models the

[redacted]

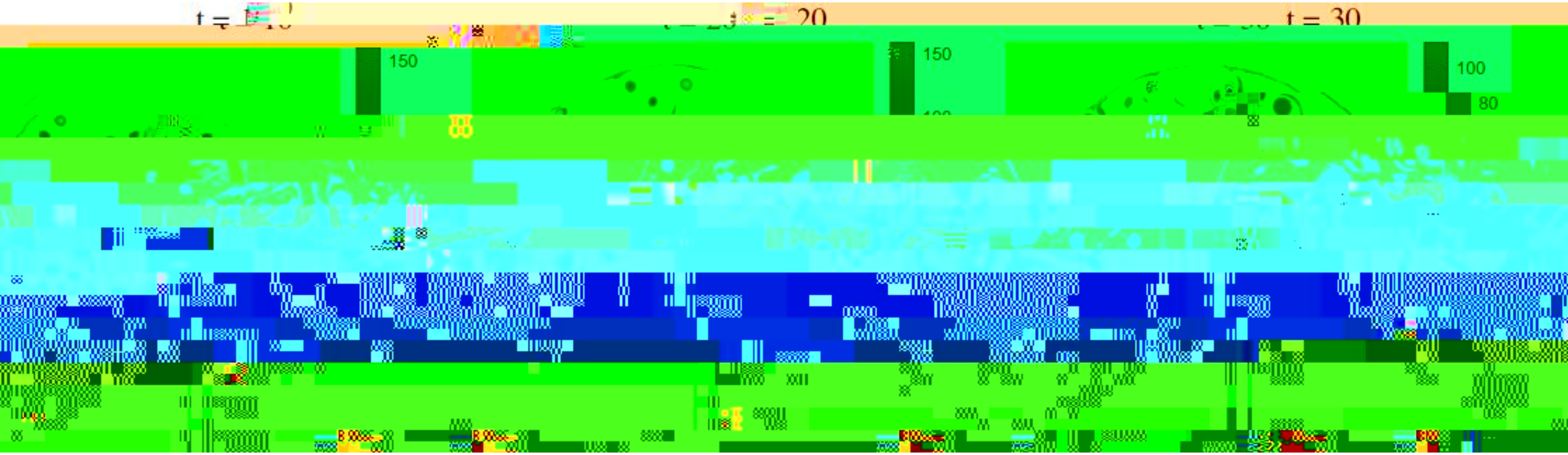
The numerical method is a Fourier pseudo-spectral method, resolution 1024^2.

Ref.: B. Kadoch, D. Kolomenskiy, P. Angot, and K. Schneider, *J. Comput. Phys.*, 231, 2012

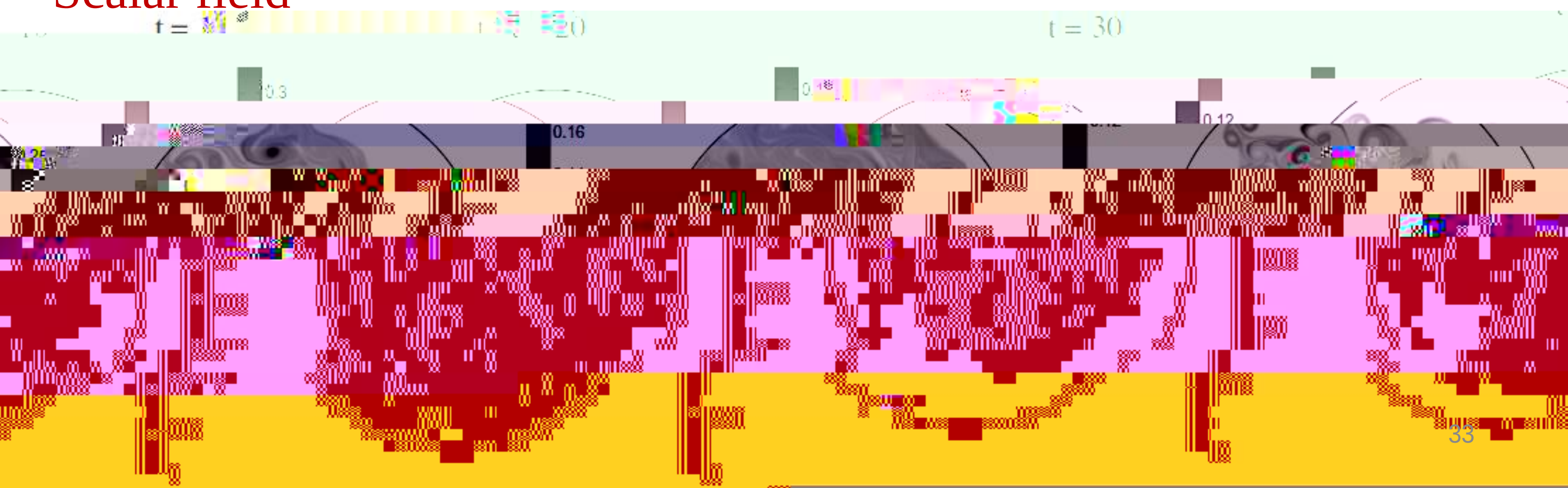
Passive scalar mixing

B. Kadoch, D. Kolomenskiy, P. Angot, K. Schneider, *JCP*, 231, 2012

Vorticity



Scalar field



The volume penalization method for MHD equations

Application to

Spontanuous rotation in toroidally confined MHD

Ref.: J. Morales, W. Bos, K. Schneider and D. Montgomery, *Phys. Rev. Lett.*, **109** , 2012.

A magnetohydrodynamic (MHD) description

MHD description ...

for the velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{B}$:

$$\frac{\partial \boldsymbol{u}}{\partial t} \; \ldots \qquad (1)$$

$$E = j - u \times B \qquad (2) \qquad\qquad \frac{\partial B}{\partial t} = -\nabla \times E$$

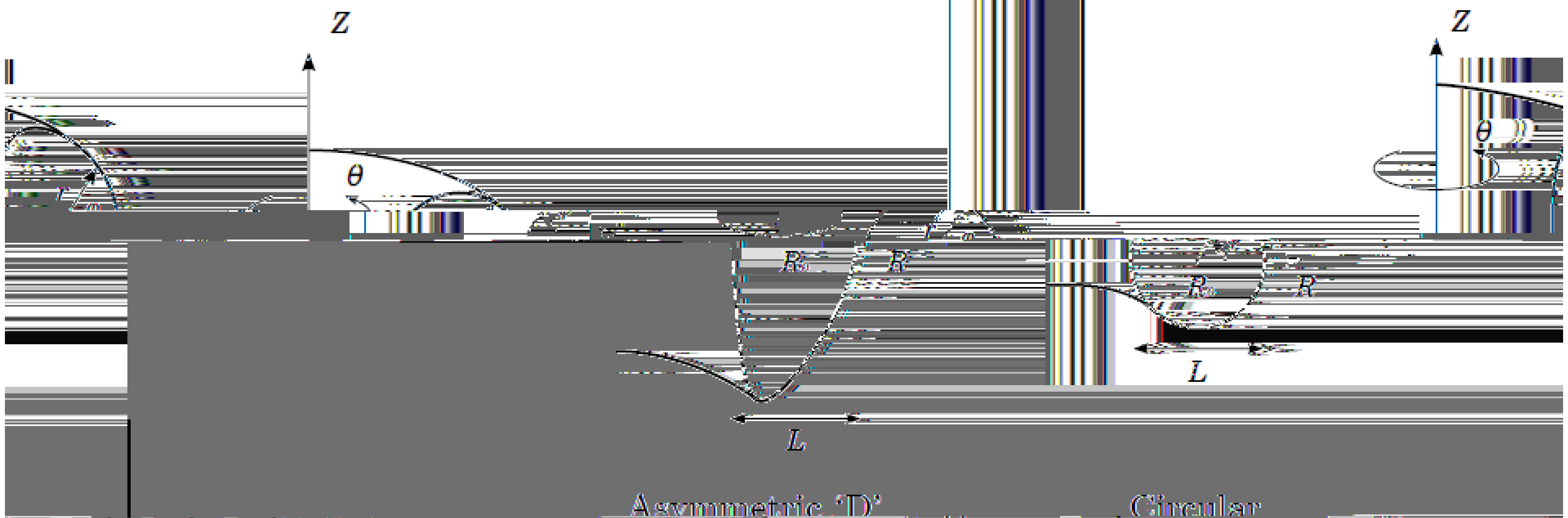
$$\ldots \qquad (3)$$

$$M = \frac{C_A L}{\nu} \qquad S = \frac{C_A L}{\lambda}$$

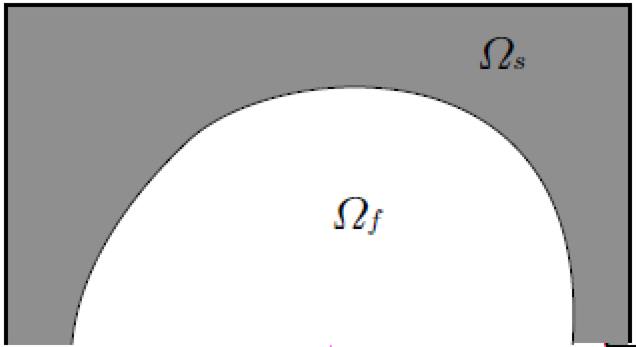
Boundary & Initial conditions (4)

(1994) and Bates & Lewis (1996) … Montgomery &

Numerical simulations



The volume penalization method



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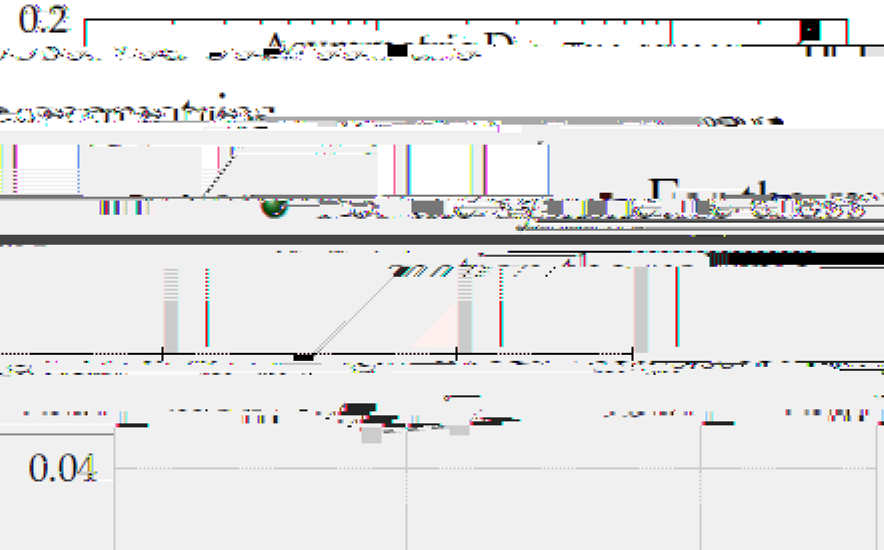
Ref.: J. Morales, W. Bos, K. Schneider and D. Montgomery, *Phys. Rev. Lett.*, **109** , 2012.

Effect of an up-down asymmetry

is observed between



For the asymmetric

averaged toroidal angular momentum

[redacted]

Ref.: J. Morales, W. Bos, K. Schneider and D. Montgomery, *Phys. Rev. Lett.*, **109** , 2012.

Effect of an up-down asymmetry

Ref.: J. Morales, W. Bos, K. Schneider and D. Montgomery, *Phys. Rev. Lett.*, **109** , 2012.

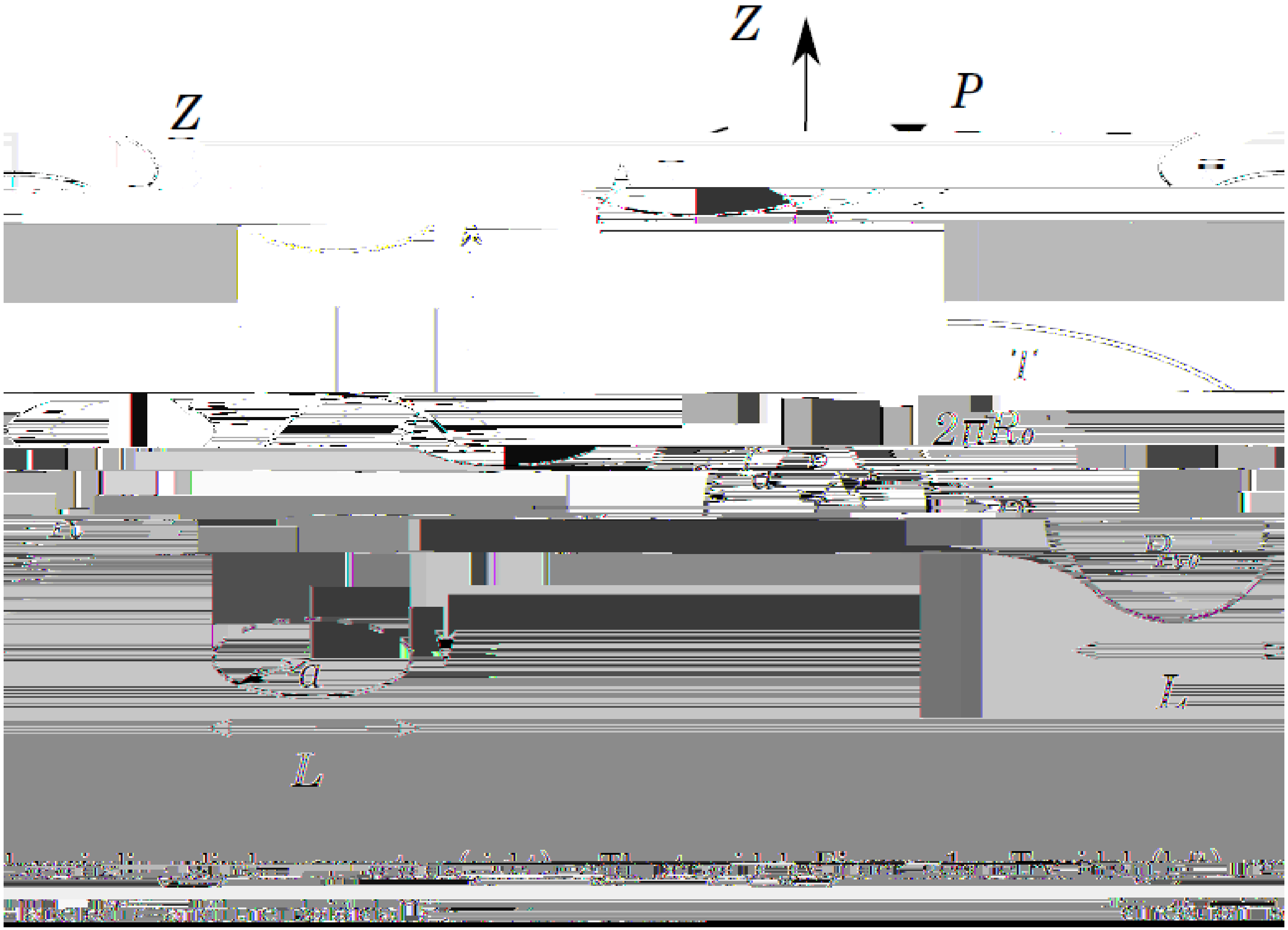
- Toroidal angular momentum is created if the up-down

Application to

Effect of toroidicity in RFP dynamics

Ref.: J. Morales, W. Bos, K. Schneider and D. Montgomery, *Plasma Phys. Control. Fusion*, **56** , 2014.

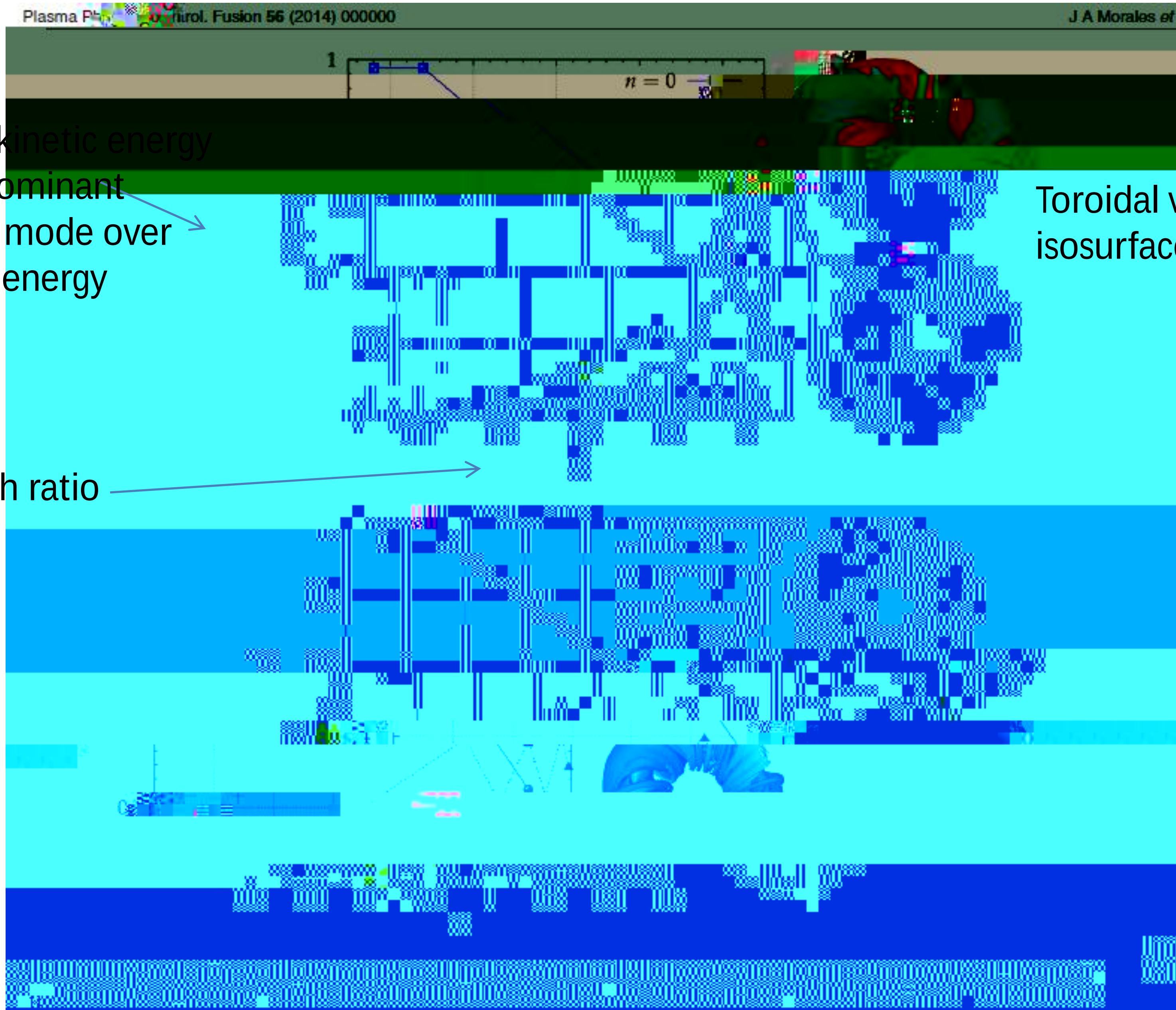
RFP in toroidal and cylinder geometry



Pinch ratio $\Theta = \overline{B_P}/\langle B_T \rangle$:
wall averaged poloidal mag. field / volume averaged toroidal mag. field

RFP dynamics (torus)

ratio of kinetic energy of the dominant toroidal mode over tot. kin. energy

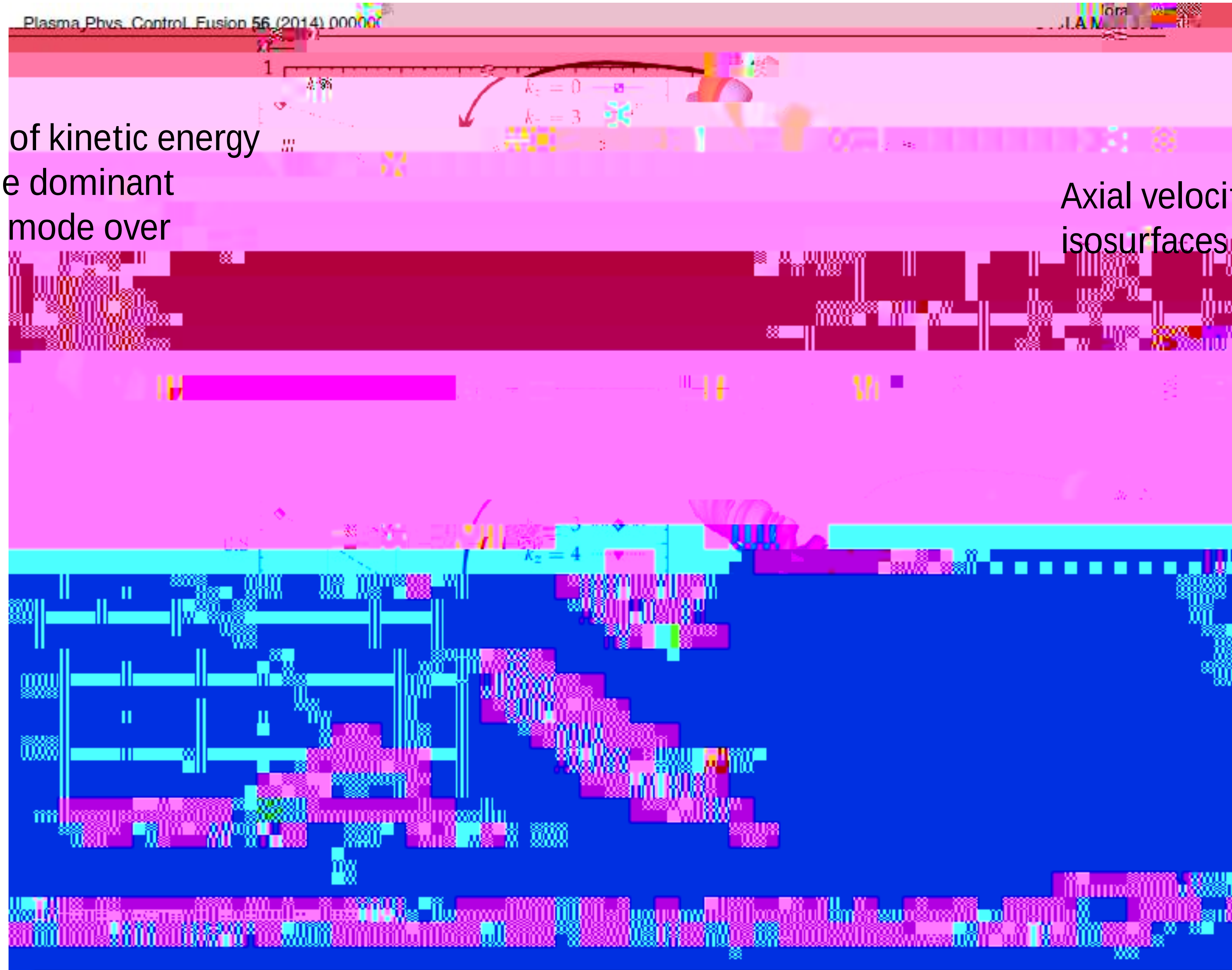


pinch ratio

Toroidal velocity isosurfaces

RFP dynamics (cylinder)

ratio of kinetic energy of the dominant axial mode over



Axial velocity isosurfaces

Conclusions: RFP dynamics

Influence of curvature of the imposed magnetic field on Reversed Field Pinch dynamics was investigated.

Comparison of a toroidal with a periodic cylindrical geometry.

Axisymmetric toroidal mode is always present in the toroidal, but absent in the cylindrical configuration.

Toroidal

Ref.: J. Morales, W. Bos, K. Schneider and D. Montgomery, *Plasma Phys. Control. Fusion*, **56** , 2014.

Conclusions

Volume penalization to model fluid and plasma flows in complex (time varying) geometries.

Simple 1d examples for the penalized Laplace with Dirichlet or Neumann boundary conditions.

Optimal penalization parameter $_{opti}$ is of order N^{-2}

The finest resolved scale should be of O($^{1/2}$)

Cancellation of penalization and discretization errors.

Applications to 2d confined fluid turbulence, fluid-

Application to plasma turbulence:

Spontaneous spin-up in toroidal geometries

Effect of toroidicity in RFP devices.

Some Theory:

Ref.: R. Nguyen van yen, D. Kolomenskiy, K. Schneider, *Numerische Mathematik*, in press, 2014 and *Appl. Num. Math.*, in press, 2014

www.cmi.univ-mrs.fr/~kschneid

Thank you very much for your attention!